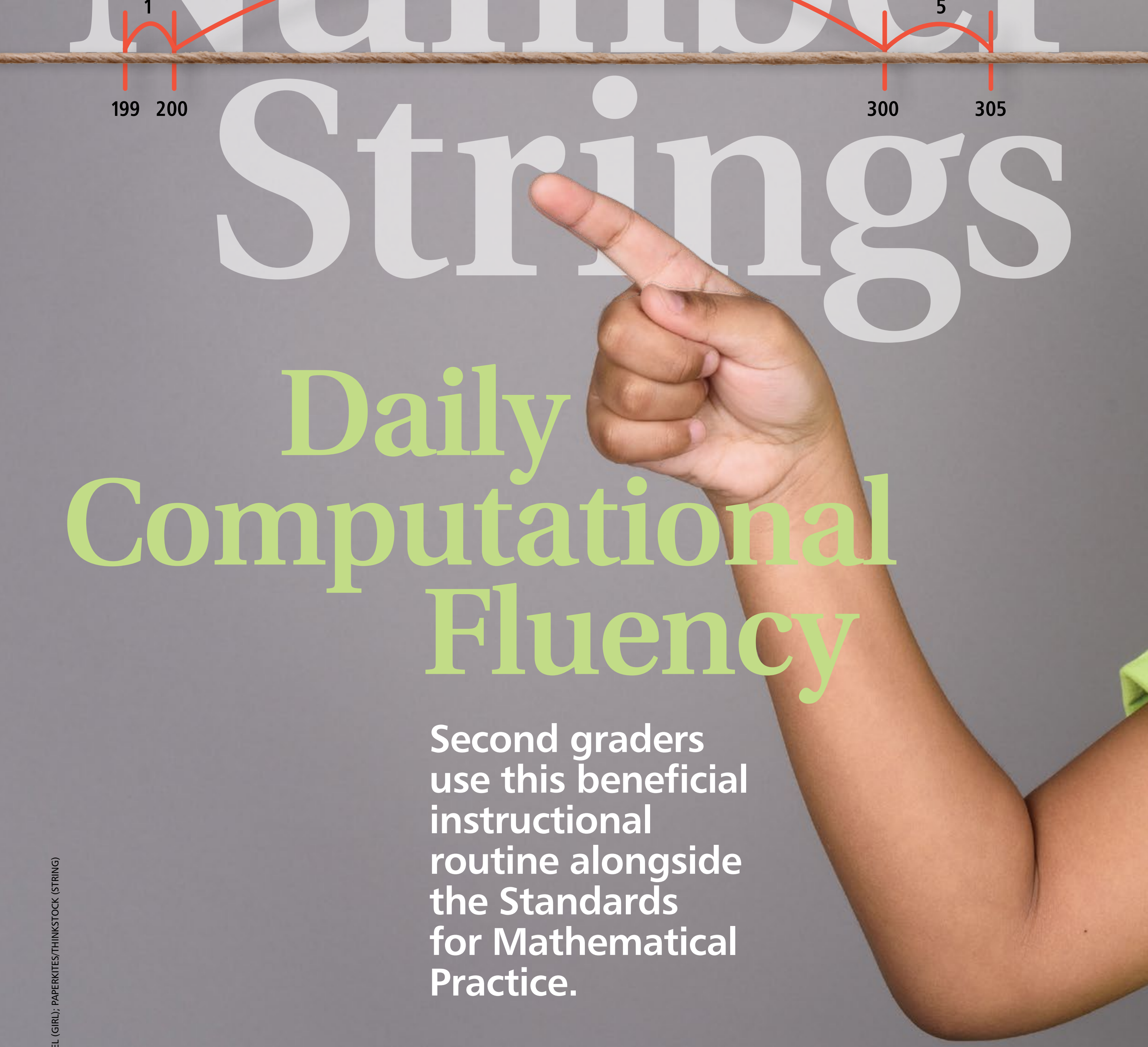
Number Strings



Daily Computational Fluency

Second graders use this beneficial instructional routine alongside the Standards for Mathematical Practice.

AMAZINGMIKAEL (GIRL); PAPERKITES/THINKSTOCK (STRING)

Rachel Lambert, Kara Imm, and Dina A. Williams

A number string is a short (15–20-minute), daily routine in which a teacher presents a carefully designed sequence of problems one at a time for children to solve mentally. It is designed so that students develop a range of strategies for mental computation, explore the connections between strategies, and deepen understanding of such mathematical models as the number line. Number strings also provide opportunities for students to engage in mathematical discourse, both in describing their own strategies and connecting with the mathematical strategies of others. Students participating in number string routines are able to adopt new strategies, supported by mathematical discourse and teacher representation of student strategies (O'Loughlin 2007). Number strings support students in making connections between conceptual understanding and procedures (Callandro 2000).

To move from conceptual understanding to procedural fluency, students need specific skills and knowledge, which the Common Core State Standards for Mathematics (CCSSM) define. Along with content standards for each grade or course, the eight Standards for Mathematical Practice (SMP) (CCSSI 2010, pp. 6–8) describe the mathematical habits of mind that teachers should aim to promote in students. A robust mathematics classroom integrates both grade-level content and mathematical practices. The daily routine of number strings is an effective instructional approach to address this "point of intersection" of mathematical knowledge and practice (Fosnot and Dolk 2002).

In this article, we illustrate how the practice of number strings—used regularly in a classroom community—can simultaneously support computational fluency and building conceptual understanding. Specifically, we will demonstrate how a lesson about multidigit addition (CCSSM 2NBT.B.5) can simultaneously serve as an invitation to look for and make use of structure (SMP 7) and engage in mathematical reasoning (SMP 2). We present the work of second-grade teacher and co-author Dina Williams and her students. Williams teaches in a school in which all students are African-American or Latino/a and almost all students are from low-income households. In addition, several students with disabilities are included in her class. An advocate of equity in mathematics, Williams aims to engage all her students in rigorous mathematics. She supports their engagement through classroom routines—in this case, daily number strings. Within this routine, she shifts mathematical authority from herself to her students, who begin to see themselves as innovators of mathematical strategies. She makes strategies explicit by naming them, which helps her students make sense of them and begin to use them. With the number string, her goal is to support her students to think flexibly and strategically about addition, with the support of the open number line as a model.

More on number strings

The mathematical content of number strings ranges from early number to rational number to algebra (see **table 1** on **p. 53**). In a typical number string, students gather at the rug or other meeting area, ready to engage in mental mathematics. After each problem is written horizontally on the board, the teacher offers significant wait time to allow students to develop a solution and strategy. A few students share strategies for each problem. Each time a new strategy or mathematical idea is shared, the teacher represents the strategy using a particular mathematical model, such as a rekenrek, an array, a ratio table, or an open number line. Although number strings are often designed to elicit particular strategies, teachers represent all student strategies (including incomplete or incorrect strategies), supporting students to think flexibly and to make sense of the relationships among strategies.

Number strings arose within realistic mathematics education (RME), a branch of mathematics education developed at the Freudenthal Institute in the Netherlands and now used internationally. RME proposes that students learn best through first engaging in problem solving in realizable (i.e., imaginable, believable) contexts, and then through strategic discussion in which students' thinking is represented by the teacher, using purposefully chosen models (Fosnot and Dolk 2002). After engaging in this process, students will be able to use models as tools for thinking as they solve mathematical problems. Number strings are integrated within RME curricula, such as Contexts for Learning Mathematics, and they can also be used as a daily routine to enhance any curriculum.

Number strings are closely related to number talks (Humphreys and Parker 2015). Both are daily routines that develop computational skills through student sharing of multiple strategies. Two primary features, however, distinguish them. First, a number talk tends to be a conversation about a single problem, with representations of a multitude of strategies. Number strings, as their name suggests, are always a sequence, or string, of related problems. Second, number talks may include such mathematical models as the open number line, but this is not an essential feature. In number strings, however, the use of particular mathematical models is central.

Designing a number string

As Williams prepared for this number string two months into the school year, she took stock of her second graders' strategies for multidigit addition. The majority of her students were using decomposition (sometimes known as *splitting*) as a strategy to add double-digit numbers, breaking numbers into place-value units and then combining those units (CCSSM 2NBT.B.5). She also had a few students who used compensation, changing a problem, for example, 59 + 12, into an equivalent expression, such as 60 + 11.

Williams wanted students to develop more strategies for addition, so she designed a number string to invite students to keep one number whole and make jumps of ten (or multiples of ten)—a relatively new strategy for her class (see **fig. 1**). Williams knew that other strategies might emerge, and she planned to represent and honor all strategies. Williams chose 33 + 10 as her first problem, establishing the place-value pattern when adding (or subtracting) tens (CCSSM 2NBT.B.8). Her next two problems (33 + 14 and 33 + 24) were designed as helper problems. They built on the first problem and each other, to suggest the strategy of keeping the first number whole and adding on groups of tens and ones. Their position in the number string encouraged students to consider the structure of the numbers—to see the tens and ones "inside" of fourteen and twenty-four and make jumps accordingly (SMP 7). The fourth problem (46 + 40) was developed to help students generalize the strategy, by breaking with the expected addend of thirty-three used in previous problems. The final problem (46 + 39) was crafted to be a challenge. By design, number strings often end with problems that initially look unfamiliar to students but that can be solved mentally with the relationships between the problems in the string or the strategies that have been developed by the class. Williams chose these numbers purposely: She wanted this final problem to invite students to use a variety of decomposition strategies, such as starting at forty-six and taking three jumps of ten and one jump of nine, or using the previous problem (46 + 40) and simply jumping back one.

Leading a number string

Williams gathered her second graders on the rug, reminding them to use a "quiet thumb" to show they were ready to share both a solution and a strategy, and making sure each student had an "elbow partner" for turn-and-talks. To support her students' transition from a number line of connecting cubes to an open number line, Williams attached a number line—built by alternating colors of connecting cubes in groups of five—to the top of her board (see **fig. 2**). Throughout the number string, she made sure that the numbers she marked on the open number line were exactly aligned to the connecting cubes above. Her careful connection of the two representations supported students in understanding magnitude of numbers on the open number line as distance. She wrote the first problem (33 + 10). Most thumbs were raised immediately. She called on a student who started at thirty-three and made a jump of ten. With some guidance from her students, she located and labeled the 33 on the line of connecting cubes, then made a jump of ten ending at the 43, and finally recorded the answer as 43. Next, she wrote the second problem (33 + 14) and again gave students "think time" to solve this problem. This time, a student started at thirty-three followed by a jump of ten and then four more jumps of one each. Williams drew a

FIGURE 1

Wanting her students to develop additional strategies, the teacher designed this number string for keeping one number whole and making jumps of ten.

33 + 10

33 + 14

33 + 24

46 + 40

46 + 39

FIGURE 2

Williams attached alternating colors of connecting cubes in groups of five to the top of her board, always carefully aligning numbers she marked on open number lines with the cubes above them.

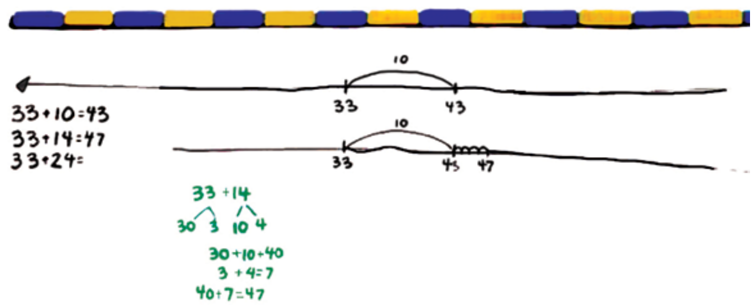
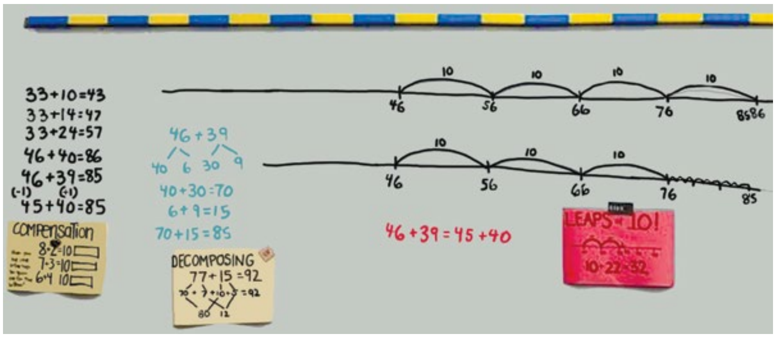


FIGURE 3

From the large-group dialogue about the last problem in the number string, 46 + 39, four student approaches arose for the five problems: compensation, two jump strategies, and decomposing.



new open number line, located and labeled the 33, drew a jump of ten to the 43, and then four more jumps of one, landing at the 47. Williams then asked if other students had solved the problem differently. A student shared a decomposition (or splitting) strategy, which Williams represented as well.

For 33 + 24, the third problem in the string (see **fig. 1**), students used both decomposition and jumps on the number line, and Williams represented both strategies. For 46 + 40, students used jumps of ten, starting at forty-six. As Williams wrote the final problem on the board, 46 + 39, a few students reacted with disbelief, "Whoa! That's big." After some individual think time, she asked students to turn and talk to a partner. In the discussion that followed, four different strategies emerged (see **fig. 3**). The first student shared his answer of seventy-five. Knowing that the answer was eighty-five, Williams surmised that he had left out a ten in his mental calculations. Without comment on the miscalculation, Williams asked him to describe his strategy. Once it was apparent that he had used decomposition, she represented his thinking using the splitting strategy (see the left side of **fig. 3**, decomposing). He broke the forty-six and the thirty-nine into tens and ones, and then first combined tens and then ones. He moved quickly through his explanation until he was recombining seventy and fifteen. He paused, looked at the representation, and then shook his head. "Eighty-five," he said, "not seventy-five." Williams asked if he was "revising" his thinking, and he agreed, nodding his head. As he talked the class through the representation, he found his own miscalculation and corrected it, as often occurs when students are encouraged to verbalize their thinking. Williams used representation as a tool to make his thinking visible so that he could reflect on it.

The second and third students both took jumps on the number line starting at forty-six, one student jumping four tens, and then one back, and the other student jumping three tens, then nine ones forward. The final student used a compensation strategy (see the far-left side of **fig. 3**), giving one from forty-six to thirty-nine, and in effect, renaming 46 + 39 as an equivalent expression of 45 + 40. Williams noted this equivalence in a number sentence. At the end of the routine, Williams noted and named the different strategies that her students had used that day (see **fig. 3**). This choice was informed both by her understanding of the mathematics as well as her mindfulness of access and equity. By making mathematical ideas explicit and public and sharing them within this community, she knew that her students had a better chance of making sense of, and later taking up, these strategies.

The power of number strings for teaching practice

Number strings are effective not only for students but also for teachers who are interested in developing their instructional practice. Guided rehearsals of number strings have helped teachers develop skills in facilitating mathematical discussion and transferring mathematical authority to students (Bofferding and Kemmerle 2015). Number strings are an instructional routine, providing opportunities to implement several Mathematical Teaching Practices from *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014, p. 10): use and connect mathematical representation, facilitate meaningful mathematical discourse, and pose purposeful questions.

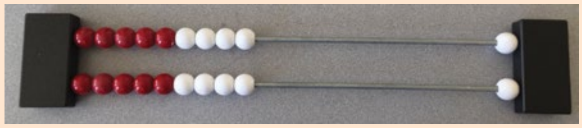
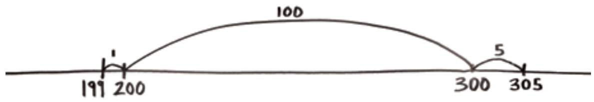
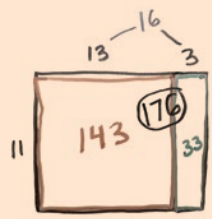
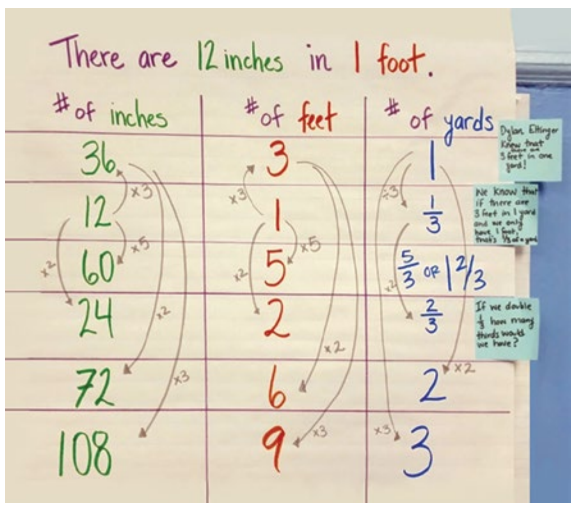
Using and connecting mathematical representations

Williams was developing her students' understanding of a mathematical model, specifically the open number line. By design, open number lines are marked only where students use them,

TABLE 1

Sample number strings show that their mathematical content runs the gamut from early number to rational number to algebra.

Sample number strings

Number string	Design and mathematical model
Early addition (doubles plus or minus one) 6 + 6 8 + 9 6 + 7 8 + 8 7 + 6 9 + 9	Number string designed to help students use doubles facts to help them with other additional problems. In this string, the teacher writes the problem and then represents student thinking using a rekenrek (also known as math rack). Developed from Fosnot and Uittenbogaard (2008a). Rekenrek 
Subtraction (comparing the strategies of adding on and removing) 82 – 9 82 – 73 112 – 6 112 – 106 305 – 199	Number string designed to help students compare adding on versus removal strategies in subtraction. Student strategies are modeled on the open number line. Developed from Fosnot and Uittenbogaard (2008b). Open number line 
Division (partial quotients) 11 × 10 143 ÷ 11 110 ÷ 10 11 × 13 110 ÷ 11 176 ÷ 11 33 ÷ 11	Number string designed to help students develop strategies for using the partial quotients in division, based on the distributive property of multiplication. Modeled on the open array. Developed from Uittenbogaard and Fosnot (2008). Open array 
Proportional thinking and ratio If 12 inches are in 1 foot, how many inches are in 3 feet? How many feet are in 60 inches? How many feet are in 72 inches? How many inches are in 9 feet? We know 3 feet are in 1 yard. If we have 1 foot, how much of a yard do we have? If we have 2 feet, how much of a yard do we have? If we have 5 feet, how many yards do we have?	Number string designed to help students convert measurements using proportional thinking structured by the ratio table. The ratio table is gradually filled, but begins empty. This representation is of the finished ratio table. Written by teacher Kathy Minas (see numberstrings.com). Ratio table 

supporting flexible and efficient strategies in addition, subtraction, multiplication, division, and algebra. Unlike other representations of number, such as base-ten blocks, the open number line's linear orientation supports students to transition from thinking exclusively about number as a quantity (e.g., 17 objects) to thinking of number as space or distance (e.g., where is 17?) (Gravemeijer 1993). Students' ability to estimate distance accurately on the number line has been correlated with increased achievement in mathematics (Siegler and Booth 2004). Williams's use of this model supported learners to visualize the connection between distance and magnitude of numbers on a number line. Anchoring computation on well-chosen models provides external scaffolding for problem solving that children gradually begin to internalize. Models are more than visuals; they are tools for children to visualize mathematical relationships. In a classroom of diverse and multilingual learners, these models provide wider access to the mathematical ideas being discussed.

Facilitate meaningful mathematical discourse

Throughout this routine, students engage in mathematical talk. Facilitating mathematical talk can be challenging for teachers, particularity novice teachers or those who are unfamiliar with standards-based mathematics practices. However, because a number string is a routine that offers consistency in structure, it presents an opportunity for teachers to develop skills in facilitating mathematical discussion, particularly when novice teachers engage in repeated guided rehearsals (Lampert et al. 2013). Research on the number string routine demonstrates its potential to deepen the level of mathematical talk in the classroom (Bofferding and Kemmerle 2015). Specific teacher strategies include using low-press questions (e.g., "How did you get that answer?") as well as high-press questions (e.g., "Will your strategy always work?") (Kazemi and Stipek 2001). Number strings provide opportunities for teachers to revoice student strategies as well as encourage students to restate the strategies of others. As teachers grow more sophisticated in their questioning, they are able to encourage students to *critique the reasoning of others*, the second of the Common Core's (CCSSI 2010, pp. 6–8) Standards for Mathematical Practice (SMP 2), in a safe and productive manner.

Build procedural fluency from conceptual understanding

As we noted earlier, Williams connected procedural fluency to conceptual ideas by exploring why strategies work. She designed number strings such as this to support students in trying new strategies as well as seeing connections among strategies. She supported strategic flexibility by naming these strategies and providing opportunities for her students to cognitively engage in the strategies of others. Once her students established a range of strategies for one operation, she designed additional number strings to explore which strategies work best for which kind of problems (see **table 1** for a subtraction example). For more suggestions on designing number strings, see **the online more4u** materials.

Conclusion

When students consistently engage in the routine of number strings, they are supported in complex mathematical thinking and discussion within the boundaries of a structured practice. As they deepen their understanding of efficient computation, they are simultaneously asked to engage in the mathematical practices: describing their thinking, comparing strategies, and justifying their reasoning to their peers. We are convinced that each and every student can engage in these practices, with the support of the number strings routine.

Common Core Connections

SMP 2
SMP 7
2.NBT.B.5

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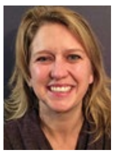

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more4U

Go to **http://www.nctm.org/tcm** for an appendix of more suggestions on designing number strings. Access is a members-only benefit.